POSITIVE SOLUTIONS OF A NONLOCAL SECOND ORDER BOUNDARY VALUE PROBLEM WITH SIGN CHANGING NONLINEARITIES

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ABSTRACT. In this paper we study a nonlocal boundary value problem concerning the equation $x''(t)+f(t,x(t),x'(t))=0,\ t\in[0,1]$. By applying fixed point theorems on appropriate cones, we prove that this boundary value problem admits positive solutions with slope in a prespecified annulus. It is remarkable that we do not assume $f\geq 0$. Here the sign of the function f may change.

1. INTRODUCTION

This paper discusses the nonlinear equation

$$x''(t)+f(t,x(t),x'(t))=0,\ \ t\in I:=[0,1] \tag{1.1}$$

associated with the initial condition

$$x(0)=0. \tag{1.2}$$

and the nonlocal boundary condition

$$x'(1)=\int_0^1 x'(s)dg(s), \tag{1.3}$$

where $f\colon I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $g\colon I\to\mathbb{R}$ is an increasing function and the integral involved in (1.3) is a Riemann-Stieltjes integral.

Condition (1.3) is obviously the continuous version of the multipoint boundary condition

$$x'(1)=\sum_{i=1}^{m}\alpha_i x'(\xi_i), \tag{1.4}$$

where $\xi_i\in(0,1)$, $i=1,...,m$ and the real numbers $\alpha_1,\alpha_2,...,\alpha_m$ have the same sign. The boundary value problem (1.1),(1.2),(1.4) appeared early in the literature

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(see [16] and the references therein) and has been recently studied among others in [13, 14]. Also the problem of the existence of positive solutions for the nonlocal boundary value problem (1.1)-(1.3) has been dealt with by Karakostas and Tsamatos in several papers [17-20]. We note that the problem of existence of positive solutions for ordinary differential equations has started from corresponding problems in partial differential equations. Indeed the problem of the existence of positive solutions for partial semilinear elliptic equations is very old, has many aplications and has been widely studied. (See the review article by Lions [23].) H. Wang in [27] showed how such a problem can be reduced to a second order boundary value problem for ordinary differential equations with two point boundary conditions. Since then, the problem of existence of positive solutions for second order ordinary differential equations, with various boundary conditions, has been dealt with by many authors. (See e.g. [2, 4-10,12,15,17-20, 22, 24-26].) The book of Ararwal, O' Regan and Wong [1] gives a good overview on this issue.

A very common assumption in the study of positive solutions for boundary value problems which concern the equation (1.1) and especially the case in which Krasnoselskii's fixed point theoren is applied, is that $f \geq 0$. (See e.g. [8-10, 15, 17-20, 22, 24-27].) This assumption is very helpful in proving that such a boundary value problem has at least one positive and concave solution. As we know there are rather few papers in which f is allowed to change sign. (See [2, 4, 5, 12, 22].) Motivated mainly by [12, 17-20, 22], in the present paper we establish sufficient conditions under which the boundary value problem (1.1)-(1.3) has positive solutions in the case which f can change sign. To obtain our results, except from the well known Krasnoselskii's fixed point theorem (see Theorem 1.2 below), we also use another fixed point theorem on a Banach space ordered by appropriate cone (see Theorem 1.1 below). The concept of the cone in a Banach space is essential to formulate and apply both these theorems.

Definition. *Let $\mathcal{B}$ be a real Banach space. A cone in $\mathcal{B}$ is a nonempty, closed set $\mathbb{K} \subset \mathcal{B}$ such that*

(i) $\kappa u + \lambda v \in \mathbb{K}$ for all $u, v \in \mathbb{K}$ and all $\kappa, \lambda \geq 0$

(ii) $u, -u \in \mathbb{K}$ implies $u = 0$.

In a Banach space $\mathcal{B}$ we denote by B_ρ the set

$$B_\rho = \{x \in \mathcal{B} : \|x\| < \rho\}$$

and by ∂B_ρ the set

$$\partial B_\rho := \{x \in \mathcal{B} : \|x\| = \rho\}.$$

Theorem 1.1. ([3. p.661]) *Let $\mathcal{B}$ a Banach space, $\mathbb{K}$ a cone in $\mathcal{B}$ and $h : \mathbb{K} \cap \overline{B_\rho} \to \mathbb{K}$ be a compact map such that $h(x) \neq \lambda x$ for all $x \in \mathbb{K} \cap \partial B_\rho$ and $\lambda \geq 1$. Then h has a fixed point in $\mathbb{K} \cap \overline{B_\rho}$.*

Theorem 1.2. ([11. 21]) *Let $\mathcal{B}$ be a Banach space and let $\mathbb{K}$ be a cone in $\mathcal{B}$. Assume that Ω_1 and Ω_2 are open bounded subsets of $\mathcal{B}$, with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let*

$$T\colon\ \mathbb{K} \cap (\overline{\Omega_2} \setminus \Omega_1) \to \mathbb{K}$$

be a completely continuous operator such that, either

$$\|Tu\| \leq \|u\|,\ u \in \mathbb{K} \cap \partial\Omega_1 \ \text{ and } \ \|Tu\| \geq \|u\|,\ u \in \mathbb{K} \cap \partial\Omega_2,$$

or

$$\|Tu\| \geq \|u\|, \ u \in \mathbb{K} \cap \partial\Omega_1 \text{ and } \|Tu\| \leq \|u\|, \ u \in \mathbb{K} \cap \partial\Omega_2.$$

Then T has a fixed point in $\mathbb{K} \cap (\overline{\Omega_2} \setminus \Omega_1)$.

2. The assumptions and the main results

In the sequel we shall denote by $\mathbb{R}$ the real line and by $\mathbb{R}^+$, I the intervals $[0, +\infty)$, $[0, 1]$ respectively. Also, by $C_0^1(I)$we denote the space of all functions $x : I \to \mathbb{R}$, whose the first derivative x' is continuous on I and such that $x(0) = 0$. We furnish the set $C_0^1(I)$ with the norm $\| \ \|_0^1$, defined by

$$\|x\|_0^1 := sup\{|x'(t)| : t \in I\},$$

to make it a Banach space.

For the function g we assume the following:

(H_1) $g: I \to \mathbb{R}$ be an increasing function such that

$$g(0) = 0 \quad \text{and} \quad g(1) \neq 1.$$

Consider equation (1.1) associated with the boundary conditions (1.2), (1.3).

By a solution of the boundary value problem (1.1) – (1.3) we mean a function $x \in C_0^1(I)$ satisfying condition (1.3) and equation (1.1) for all $t \in I$.

Searching for the existence of solutions we shall reformulate the problem (1.1) – (1.3) to obtain an operator equation of the form $x = Ax$, for an appropriate operator A. To find A consider an equation of the form

$$x''(t) + z(t) = 0, \quad t \in I,$$

subject to conditions (1.2), (1.3). By integration we get

$$x'(t) = x'(1) + \int_t^1 z(s)ds. \tag{2.1}$$

Then, from (1.3), it follows that

$$x'(1) = \gamma \int_0^1 \int_t^1 z(s)dsdg(t),$$

where

$$\gamma := \frac{1}{1 - g(1)}.$$

Integrating once again (2.1) we obtain

$$x(t) = \gamma t \int_0^1 \int_s^1 z(r)drdg(s) + \int_0^t \int_s^1 z(r)drds, \ t \in I.$$

This process shows that solving the boundary value problem (1.1) – (1.3) is equivalent to solve the operator equation $x = Ax$ in $C_0^1(I)$, where A is the operator defined by

$$(Ax)(t) = \gamma t \int_0^1 \int_s^1 f(r, x(r), x'(r))drdg(s) + \int_0^t \int_s^1 f(r, x(r), x'(r))drds, \tag{2.2}$$

for $x \in C_0^1(I)$ and $t \in I$.

Before presenting our results we give our assumptions as well as some useful notations as follows:

(H_2) There exist nonnegative real valued functions p, q, r in $L^1(I)$ and Φ, Ψ nonnegative, nondecreasing real valued functions, locally integrable on $\mathbb{R}^+$ and such that

$$|f(t,u,v)| \leq p(t) + q(t)\Phi(|u|) + r(t)\Psi(|v|),$$

for all $(t,u,v) \in I \times \mathbb{R} \times \mathbb{R}$. Moreover, we assume that the functions p, q and r are not equal to zero almost everywhere.

Next define the continuous functions

$$P(t) := \int_t^1 p(s)ds, \;\; Q(t) := \int_t^1 q(s)ds, \;\; R(t) := \int_t^1 r(s)ds, \;\; t \in I,$$

$$Q_m(t) := \int_t^1 q(\theta)\Phi(\theta m)d\theta, \;\; t \in I, \;\; m \in \mathbb{R}^+$$

and

$$F(m) := \gamma \int_0^1 \int_s^1 f(r, mr, m)drdg(s), \;\; m \in \mathbb{R}^+.$$

Moreover, if $w : I \to \mathbb{R}$ is a continuous function, by the symbol w_g we define the number

$$w_g = \int_0^1 w(s)dg(s).$$

(H_3) Assume that there exists $T > 0$ such that

$$|\gamma|(P_g + (Q_T)_g + \Psi(T)R_g) + P(0) + Q_T(0) + R(0)\Psi(T) < T.$$

(H_4) The function $\gamma f(t,u,v)$ is non-increasing with respect to the variables u, v.

Now we are in position to prove our first main result:

Theorem 2.1. *Let the assumptions $(H_1) - (H_4)$ be satisfied and, moreover,*

(H_5) *there exists $\Theta > T$ such that $F(\Theta) - P(0) - Q_\Theta(0) - \Psi(\Theta)R(0) \geq 0$.*

Then there exists a solution $x \in C_0^1(I)$ of the boundary value problem (1.1) – (1.3) such that $0 \leq x(t) \leq Tt$, $t \in I$, where T is defined by (H_3).

Proof. Our purpose is to make the appropriate arrangements so that Theorem 1.1 to be applicable. For this purpose consider the set

$$\mathbb{K} := \{x \in C_0^1(I) : x'(t) \geq 0, \;\; t \in I\},$$

which is a cone in $C_0^1(I)$ and the set

$$B_\Theta = \{x \in C_0^1(I) : \|x\|_0^1 < \Theta\},$$

where Θ is the positive constant ensured by assumption (H_5). We will prove that

$$A(\mathbb{K} \cap \overline{B_\Theta}) \subset \mathbb{K}.$$

Indeed, if $x \in \mathbb{K} \cap \overline{B_\Theta}$, then $0 \leq x'(t) \leq \Theta$ and $0 \leq x(t) \leq \Theta t$, $t \in I$. Moreover, by (H_2), (H_4) and (H_5), for every $t \in I$ we have

$$\begin{aligned}
(Ax)'(t) &= \gamma \int_0^1 \int_s^1 f(\theta, x(\theta), x'(\theta)) d\theta dg(s) + \int_t^1 f(\theta, x(\theta), x'(\theta)) d\theta \\
&\geq \gamma \int_0^1 \int_s^1 f(\theta, x(\theta), x'(\theta)) d\theta dg(s) - \int_t^1 |f(\theta, x(\theta), x'(\theta))| d\theta \\
&\geq \gamma \int_0^1 \int_s^1 f(\theta, \Theta\theta, \Theta) d\theta dg(s) \\
&\quad - \int_0^1 \left(p(\theta) + q(\theta)\Phi(|x(\theta)|) + r(\theta)\Psi(|x'(\theta)|)\right) d\theta \\
&\geq F(\Theta) - \int_0^1 \left(p(\theta) + q(\theta)\Phi(\theta\Theta) + r(\theta)\Psi(\Theta)\right) d\theta \\
&\geq F(\Theta) - P(0) - Q_\Theta(0) - \Psi(\Theta)R(0) \geq 0.
\end{aligned} \tag{2.3.}$$

Now we define the open set

$$B_T = \{x \in C_0^1(I) : \|x\|_0^1 < T\},$$

where T is the positive constant ensured by assumption (H_3). Since, by (H_5), $T < \Theta$, we have $\mathbb{K} \cap B_T \subset \mathbb{K} \cap B_\Theta$ and thus

$$A(\mathbb{K} \cap B_T) \subset \mathbb{K}.$$

Furthermore, we will show that $\lambda x \neq Ax$, for every $\lambda \geq 1$ and $x \in \mathbb{K}$, with $\|x\|_0^1 = T$. So, let $x \in \mathbb{K}$, with $\|x\|_0^1 = T$ and $\lambda \in [1, +\infty)$ such that $\lambda x = Ax$. Then $0 \leq x'(t) \leq T$ and $0 \leq x(t) \leq Tt$, $t \in I$. Thus, taking into account assumption (H_2), for every $t \in I$ we have

$$\begin{aligned}
x'(t) &\leq \lambda x'(t) = (Ax)'(t) \\
&\leq |\gamma| \int_0^1 \int_s^1 |f(\theta, x(\theta), x'(\theta))| d\theta dg(s) + \int_t^1 |f(\theta, x(\theta), x'(\theta))| d\theta \\
&\leq |\gamma| \int_0^1 \int_s^1 \left(p(\theta) + q(\theta)\Phi(x(\theta)) + r(\theta)\Psi(x'(\theta))\right) d\theta dg(s) \\
&\quad + \int_0^1 \left(p(\theta) + q(\theta)\Phi(x(\theta)) + r(\theta)\Psi(x'(\theta))\right) d\theta \\
&\leq |\gamma| P_g + |\gamma| \int_0^1 \int_s^1 q(\theta)\Phi(\theta T) d\theta dg(s) + |\gamma| \int_0^1 \int_s^1 r(\theta)\Psi(T) d\theta dg(s) \\
&\quad + P(0) + \int_0^1 q(\theta)\Phi(\theta T) d\theta + \int_0^1 r(\theta)\Psi(T) d\theta \\
&= |\gamma|(P_g + Q_{T_g} + \Psi(T)R_g) + P(0) + Q_T(0) + R(0)\Psi(T)
\end{aligned} \tag{2.4.}$$

Therefore we have

$$T = \|x\|_0^1 = \sup_{t\in I}|x'(t)| \leq |\gamma|(P_g + Q_{T_g} + \Psi(T)R_g) + P(0) + Q_T(0) + R(0)\Psi(T),$$

which contradicts to (H_3).

The above statements ensure that Theorem 1.1 is applicable and the proof of our theorem follows. □

In order to prove the second our main result we need the following lemma.

Lemma 2.2. *Let the assumptions $(H_1) - (H_3)$ be satisfied and, moreover,*

$$(H_6) \qquad F(0) - P(0) - Q(0)\Phi(0) - R(0)\Psi(0) > 0,$$

Then there exists there exists a number $M \in (0,T)$ such that

$$(2.5) \qquad F(M) = P(0) + Q_M(0) + R(0)\Psi(M) + M,$$

where T is defined by (H_3).

Proof. Define the continuous real valued function,

$$L(m) := F(m) - P(0) - Q_m(0) - R(0)\Psi(m) - m, \quad m \geq 0.$$

By assumption (H_6) we have $L(0) > 0$. Therefore, since F and L are continuous, it is enough to show that

$$L(T) < 0.$$

To this end observe that

$$\begin{aligned}
F(T) &= \gamma \int_0^1 \int_s^1 f(\theta, \theta T, T)d\theta dg(s) \\
&\leq |\gamma| \int_0^1 \int_s^1 |f(\theta, \theta T, T)|d\theta dg(s) \\
&\leq |\gamma| \int_0^1 \int_s^1 (p(\theta) + q(\theta)\Phi(\theta T) + r(\theta)\Psi(T))\, d\theta dg(s) \\
&\leq |\gamma| P_g + |\gamma| \int_0^1 \int_s^1 q(\theta)\Phi(\theta T)d\theta dg(s) + |\gamma| \int_0^1 \int_s^1 r(\theta)\Psi(T)d\theta dg(s) \\
&= |\gamma|(P_g + Q_{T_g} + R_g\Psi(T)).
\end{aligned}$$

Hence it suffices to prove that

$$|\gamma|(P_g + Q_{T_g} + R_g\Psi(T)) - P(0) - Q_T(0) - R(0)\Psi(T) - T < 0,$$

which, in view of (H_3), is obvious.

□

Theorem 2.3. *Let the assumptions* $(H_1) - (H_6)$ *be satisfied. Then there exists a solution* $x \in C_0^1(I)$ *of the boundary value problem* (1.1) – (1.3) *such that* $M \leq \|x\|_1^0 \leq T$, *where* M *and* T *are defined by* (2.5) *and* (H_3), *respectively.*

Proof. Our purpose is to apply Theorem 1.2. Indeed, consider the cone $\mathbb{K}$ and the set B_Θ as in the proof of Theorem 2.1 and define the open sets

$$\Omega_1 := B_M = \{x \in C_0^1(I) : \|x\|_0^1 < M\} \text{ and } \Omega_2 := B_T = \{x \in C_0^1(I) : \|x\|_0^1 < T\}.$$

Let also A be the operator defined by (2.2) on the cone $\mathbb{K}$. As in the proof of Theorem 2.1, we can prove that

$$A(\mathbb{K} \cap \overline{B_\Theta}) \subset \mathbb{K}.$$

Since $M < T < \Theta$, we have $\mathbb{K} \cap (\overline{\Omega_2} \setminus \Omega_1) \subset \mathbb{K} \cap \overline{B_\Theta}$ and thus

$$A(\mathbb{K} \cap (\overline{\Omega_2} \setminus \Omega_1)) \subset \mathbb{K}.$$

Now consider a $x \in \mathbb{K} \cap \partial\Omega_1$. Then $\|x\|_0^1 = M$ and, following the same arguments as in (2.3), we derive

$$(Ax)'(t) \geq F(M) - P(0) - Q_M(0) - \Psi(M)R(0).$$

Therefore, taking into account (2.5) we have

$$(Ax)'(t) \geq M = \|x\|_0^1, \ \ t \in I,$$

which means that

$$\|Ax\|_0^1 \geq \|x\|_0^1, \ \ x \in \mathbb{K} \cap \partial\Omega_1.$$

Now consider a point $x \in \mathbb{K} \cap \partial\Omega_2$ Then $\|x\|_0^1 = T$ and, following the same arguments as in (2.4) and taking into account (H_3), we obtain

$$\|Ax\|_0^1 = \sup_{t \in I} |(Ax(t))'| \leq |\gamma| P_g + P(0)|\gamma| Q_g(T) + Q_T(0) + \Psi(T)(|\gamma| R_g + R(0)) \leq T,$$

i.e.

$$\|Ax\|_0^1 \leq \|x\|_0^1, \ \ x \in \mathbb{K} \cap \partial\Omega_2.$$

The above statements ensure that Theorem 1.2 is applicable and the proof of our theorem follows. □

3. Applications

The sublinear case. Here suposse that f satisfies the following condition:

$(\widehat{H}_2)$ There exist nonnegative real valued functions p, q, r in $L^1(I)$, not equal to zero almost everywhere and such that

$$|f(t, u, v)| \leq p(t) + q(t)|u| + r(t)|v|,$$

for all $u, v \in \mathbb{R} \times \mathbb{R}$ all $t \in I$.

This is, obviously, the case of assumption (H_2) when Φ and Ψ are both the identity functions. Then we have that

$$Q_m(t) = m\int_t^1 \theta q(\theta)d\theta \text{ and } Q_{m_g} = m\int_0^1\int_s^1 \theta q(\theta)d\theta dg(s)$$

and thus assumption (H_3) takes the form:

$(\widehat{H}_3)$ There exists $T > 0$ such that

$$|\gamma|(P_g + T\int_0^1\int_s^1 \theta q(\theta)d\theta dg(s) + TR_g) + P(0) + T\int_0^1 \theta q(\theta)d\theta + TR(0) < T. \tag{3.1}$$

Now we set

$$\widehat{q}(t) = tq(t) \text{ and } \widehat{Q}(t) = \int_t^1 \widehat{q}(s)ds$$

and by (3.1) we obtain

$$T(1 - |\gamma|\widehat{Q}_g - |\gamma|R_g - \widehat{Q}(0) - R(0)) > |\gamma|P_g + P(0).$$

Then, if

$$|\gamma|(\widehat{Q}_g + R_g) + \widehat{Q}(0) + R(0) < 1, \tag{3.2}$$

we can take

$$T > \frac{|\gamma|P_g + P(0)}{1 - |\gamma|(\widehat{Q}_g + R_g) - \widehat{Q}(0) - R(0)} =: K. \tag{3.3}$$

Moreover assumptions (H_5) and (H_6) become:

$(\widehat{H}_5)$ there exists $\Theta > K$ such that $F(\Theta) - P(0) - Q_\Theta(0) - \Theta R(0) \geq 0$,

and

$(\widehat{H}_6)$ $$F(0) > P(0),$$

respectively.

Therefore we have the following corollaries of Theorems 2.1. and 2.3. respectively.

Corollary 3.1. *Let the assumptions $(H_1), (\widehat{H}_2), (H_4), (\widehat{H}_5)$ be satisfied. and moreover (3.2) holds. Then there exists a solution $x \in C_0^1(I)$ of the boundary value problem (1.1) – (1.3) such that $0 \leq x(t) \leq Tt$, $t \in I$, where $T > K$ and K is defined by (3.3).*

Corollary 3.2. *Let the assumptions* $(H_1), (\widehat{H}_2), (H_4), (\widehat{H}_5), (\widehat{H}_6)$ *be satisfied. and moreover (3.2) holds. Then there exists a solution* $x \in C_0^1(I)$ *of the boundary value problem* (1.1) – (1.3) *such that* $M \leq \|x\|_1^0 \leq T$, *where* $T > K$ *and* K, M *are defined by (3.3), (2.5), respectively.*

An example. Consider the following nonlocal boundary value problem:

$$x''(t) + bx'(t) + btx(t) - 1 = 0, \quad t \in I, \tag{3.4}$$

$$x(0) = 0, \quad x'(1) = \int_0^1 x'(s)dg(s), \tag{3.5}$$

where b is a positive constant and $g(s) = \frac{5}{4}s^2$.

Our purpose is to check the applicability or the Corollary 3.2. to this boundary value problem. In the present case we have $f(t,u,v) = -1 + btu + bv$ and thus $|f(t,u,v)| \leq 1 + b|u| + b|v|$. Then it is clear that assumptions $(H_1), (H_2)$ are satisfied and, moreover, $p(t) = 1$ and $q(t) = r(t) = b$. Furthermore we have

$$\gamma = -4, \quad F(0) = -4\int_0^1 \int_s^1 (-1)d\theta dg(s) = \frac{5}{3} \quad \text{and} \quad P(0) = \int_0^1 p(s)ds = 1.$$

Observe that $F(0) = \frac{5}{3} > 1 = P(0)$, namely assumption $(\widehat{H}_6)$ is satisfied. Also, since $\gamma f(t,u,v) = 4 - 4btu - 4bv$, γf is nonincreasing with respect to the variables u, v, i.e. assumption (H_4) is satisfied. Moreover we have

$$\widehat{Q}(t) = \int_t^1 bsds = \frac{1-t^2}{2}b, \qquad \widehat{Q}_g = \int_0^1 \widehat{Q}(s)dg(s) = \frac{5b}{16},$$

$$R(t) = (1-t)b, \quad R(0) = b, \quad \text{and} \quad R_g = \frac{5b}{12}.$$

Then, inequality (3.2) takes the form $b < \frac{12}{53}$. Moreover we have

$$P(t) = 1 - t, \quad P_g = \frac{5}{12}$$

and thus, since $b < \frac{12}{53}$, by (3.3) we obtain

$$T > \frac{32}{12 - 53b}.$$

It remains to prove that assumptions $(\widehat{H}_3), (\widehat{H}_5)$ hold. To do this we observe that

$$F(\Theta) = \gamma \int_0^1 \int_s^1 (-1 + b\theta^2\Theta + b\Theta)d\theta dg(s) = \frac{5 - 8b\Theta}{3},$$

$$P(0) = 1, \quad Q_\Theta(0) = \frac{b\Theta}{2} \quad \text{and} \quad \Psi(\Theta) = \Theta.$$

Hence $(\widehat{H}_5)$ becomes:

$$\text{there exists } \Theta > K = \frac{32}{12 - 53b} \text{ such that } \Theta \leq \frac{4}{25b}.$$

Finally, using (2.5), we can investigate that $M = \frac{4}{25b+6}$.

Since we must have $M < T < \Theta$, we must also have

$$\frac{4}{25b+6} = M \leq \frac{32}{12 - 53b} < T < \Theta \leq \frac{4}{25b}.$$

It is clear that this inequality can hold if $0 < b < \frac{48}{1022}$. In conclusion, if $0 < b < \frac{48}{1022}$, all arrangements of Corollary 3.2. are fulfilled and thus we proved the following result:

If $0 < b < \frac{48}{1022}$, there exists a solution $x \in C_0^1(I)$ of the boundary value problem (3.4), (3.5) such that $\frac{4}{25b+6} \leq \|x\|_1^0 < \frac{4}{25b}$.

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